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The doublet representation of non-Hilbert eigenstates of the Hamiltonian. II

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Applying the doublet representation we analyze the solutions of a Hamiltonian system which has eigenstates with complex eigenvalues. The example of the Friedrichs model allows us to show how the appearance of solutions with non-Hilbert initial conditions is linked to the energy degeneration of the Hamiltonian spectrum. We discuss the difficulties of giving a physical meaning to the growing or decaying non-Hilbert solutions. We also suggest a way to circumvent the problem of the anomalous probabilities related to both complex energy eigenvalues and degeneration of the spectrum.
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I. INTRODUCTION

The doublet representation[1] offers a useful way to study the dynamical evolution of both Hilbert and non-Hilbert solutions of a Hamiltonian system. In the scheme of this formalism any physical system is represented, not only by its wave functions $\varphi(\omega)$ but also by a partner $\varphi^{\star}(\omega)$. These two functions allow us to construct an invariant scalar that, in case of Hilbert states, becomes the probability of the appearance of the corresponding state. For non-Hilbert solutions, the product of the wave function and its partner remains an invariant scalar, but its probabilistic interpretation is under discussion. Precisely in this work we apply the doublet representation to show the difficulties of giving a physical interpretation to non-Hilbert solutions and we analyze the relation between energy degeneration and the appearance of complex eigenvalues. The explicit computations performed in the frame of the Friedrichs model allow us to study the conditions that the interaction Hamiltonian must satisfy to keep the spectrum in the real domain. In the other case we show how to circumvent the problem of badly behaved probabilities by means of defining a ''reduced'' space that solves the degeneration problem.

The paper is organized as follows: in Sec. II we make a brief review of Ref. 1. In Sec. III we analyze which initial conditions correspond to states in Hilbert space $\mathscr{H}$ and which ones correspond to non-Hilbert states. Section IV is devoted to the study of the anomalous behavior of probabilities and mean values for non-Hilbert states. The conditions that the interaction must satisfy in order to obtain a real spectrum are studied in Sec. V. In Sec. VI we find a basis of the reduced space where the energy degeneration is removed. Finally, in Sec. VII we draw our main conclusions.

II. THE DOUBLET REPRESENTATION

Hamiltonian equations have, in general, well-defined continuous solutions $\varphi(\omega)$ that may or may not belong to $\mathscr{H}$. But in both cases it is possible to define a ''partner'' of the wave function, namely $\varphi^{\star}(\omega)$, that is not necessarily the complex conjugate of $\varphi(\omega)$.[1] Then, solutions of the Hamiltonian equations will be represented by the ''doublet'' $(\varphi,\varphi^{\star})$. In the particular case where

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$\varphi \in \mathscr{H}$, solutions satisfy $\varphi^\star = \varphi^*$, where φ^* is the complex conjugate of φ. In Ref. 1 we have applied this formalism to study the Friedrichs Hamiltonian and we have obtained that the field equations in the doublet representation read as

$$\omega_0 \varphi_1 + \lambda \int_0^\infty g(\omega)\varphi(\omega) d\omega = i\dot{\varphi}_1, \tag{1}$$

$$\omega \varphi(\omega) + \lambda g(\omega) \varphi_1 = i \dot{\varphi}(\omega), \tag{2}$$

$$\omega_0 \varphi_1^\star + \lambda \int_0^\infty g(\omega)\varphi^\star(\omega) d\omega = -i\dot{\varphi}_1^\star, \tag{3}$$

$$\omega \varphi^\star(\omega) + \lambda g(\omega) \varphi_1^\star = -i \dot{\varphi}^\star(\omega), \tag{4}$$

where $\varphi_1, \varphi_1^\star, \varphi(\omega), \varphi^\star(\omega)$ are the wave functions of the doublet representation (the first two for the discrete mode), $\lambda \in \mathfrak{R}$ is the coupling constant, $\omega_0 \in \mathfrak{R}^+$ is the discrete eigenvalue of the free Hamiltonian, $0 \leqslant \omega < \infty$ is the continuous spectrum, and $g(\omega) = g^*(\omega)$ stands for the interaction function.

We demand that wave functions satisfy the following natural conditions:

$$\varphi_1 \varphi_1^\star + \int_0^\infty \varphi(\omega)\varphi^\star(\omega) d\omega < \infty, \tag{5}$$

$$(\varphi_1^\star)^\star = \varphi_1, \quad [\varphi^\star(\omega)]^\star = \varphi(\omega). \tag{6}$$

With these conditions, the solution to system (1)–(4) is

$$\varphi_1(t) = \frac{1}{\sqrt{\alpha'(z_0)}} \tilde{\varphi}_1 e^{-iz_0 t} + \int_0^\infty \frac{\lambda g(\tilde{\omega})}{\alpha(\tilde{\omega})} \tilde{\varphi}(\tilde{\omega}) e^{-i\omega t} d\tilde{\omega}, \tag{7}$$

$$\varphi_1^\star(t) = \frac{1}{\sqrt{\alpha'(z_0)}} \tilde{\varphi}_1^\star e^{+iz_0 t} + \int_0^\infty \frac{\lambda g(\tilde{\omega})}{\alpha(\tilde{\omega})} \tilde{\varphi}^\star(\tilde{\omega}) e^{+i\omega t} d\tilde{\omega}, \tag{8}$$

$$\varphi(\omega,t) = \frac{1}{\sqrt{\alpha'(z_0)}} \frac{\lambda g(\omega)}{z_0 - \omega} \tilde{\varphi}_1 e^{-iz_0 t} + \tilde{\varphi}(\omega) e^{-i\omega t} + \int_0^\infty \frac{\lambda^2 g(\omega) g(\tilde{\omega})}{(\tilde{\omega} - \omega)\alpha(\tilde{\omega})} \tilde{\varphi}(\tilde{\omega}) e^{-i\tilde{\omega} t} d\tilde{\omega}, \tag{9}$$

$$\varphi^\star(\omega,t) = \frac{1}{\sqrt{\alpha'(z_0)}} \frac{\lambda g(\omega)}{z_0 - \omega} \tilde{\varphi}_1^\star e^{+iz_0 t} + \tilde{\varphi}^\star(\omega) e^{+i\omega t} + \int_0^\infty \frac{\lambda^2 g(\omega) g(\tilde{\omega})}{(\tilde{\omega} - \omega)\alpha(\tilde{\omega})} \tilde{\varphi}^\star(\tilde{\omega}) e^{+i\tilde{\omega} t} d\tilde{\omega}, \tag{10}$$

where

$$z_0(\lambda) = \omega_0 + \lambda^2 \int_0^\infty \frac{g^2(\omega)}{(z_0(\lambda) - \omega)} d\omega, \tag{11}$$

$$\alpha(z) = \omega_0 - z - \lambda^2 \int_0^\infty \frac{g^2(\omega)}{z - \omega} d\omega, \tag{12}$$

$\alpha' = d\alpha/dz$, and $(\tilde{\varphi}_1, \tilde{\varphi}(\tilde{\omega}))$, $(\tilde{\varphi}_1^\star, \tilde{\varphi}^\star(\tilde{\omega}))$ are the eigenfunctions of the free plus the interaction Hamiltonians. The singularities in $\tilde{\alpha}(\tilde{\omega})$ and $(\tilde{\omega} - \omega)^{-1}$ must be avoided making the shift $\pm i\epsilon$. We do not write it explicitly in order not to complicate the notation. A main characteristic of the solutions (7)–(10) is that they behave in a continuous way when the coupling constant λ goes to zero.

When $(\varphi_1, \varphi(\omega))$ belongs to $\mathscr{H}$, we have

$$\varphi_1^\star = \varphi_1^* \quad \text{and} \quad \varphi^\star(\omega) = \varphi^*(\omega).$$

In this particular case, Eq. (5) implies that

$$0 \leqslant \varphi_1 \varphi_1^\star \leqslant 1,$$

$$0 \leqslant \int_{\omega_1}^{\omega_2} \varphi(\omega) \varphi^*(\omega) d\omega \leqslant 1.$$

If $[\omega_1, \omega_2]$ is any nonempty interval, the last integral corresponds to a well-defined probability.

III. NON-HILBERT INITIAL CONDITIONS

From Eqs. (7)–(10) we can see that $\varphi^\star$ equals φ^* when $z_0 \in \Re$. In this case, the solutions belong to $\mathscr{H}$. But as we can obtain Hilbert solutions, even for those Hamiltonians that have complex eigenvalues, we are now interested in determining, for nonreal z_0, which initial conditions correspond to those states.

Initial conditions in $\mathscr{H}$ imply that $\varphi_1^\star(t=0) = \varphi_1^*(t=0)$ and $\varphi^\star(\omega, t=0) = \varphi^*(\omega, t=0)$. When analyzing the solutions (7)–(10) to the equations of motion, we see that this relation of conjugation holds true for any time value if the first terms of the rhs in (7)–(10) cancel with the corresponding residue evaluated in z_0. It is easy to see that these four conditions reduce to the following two:

$$2\pi i \ \mathrm{Res}\left[\frac{\lambda g(\omega)}{\alpha(\omega)} \ \tilde{\varphi}(\omega), z_0\right] = -\frac{1}{\sqrt{\alpha'(z_0)}} \ \tilde{\varphi}_1, \tag{13}$$

$$2\pi i \ \mathrm{Res}\left[\frac{\lambda g(\omega)}{\alpha(\omega)} \ \tilde{\varphi}^\star(\omega), z_0\right] = -\frac{1}{\sqrt{\alpha'(z_0)}} \ \tilde{\varphi}_1^\star. \tag{14}$$

An illustrative example with initial conditions in $\mathscr{H}$ that satisfy (13) and (14) is the general state whose initial conditions coincide with the discrete eigenstate of the free Hamiltonian H_0:

$$\varphi_1(t=0) = \varphi_1^\star(t=0) = 1, \quad \varphi(\omega, t=0) = \varphi^\star(\omega, t=0) = 0,$$

and evolve with the complete Hamiltonian H. Indeed, performing the change of basis (7)–(10), we obtain that

$$\tilde{\varphi}(\omega) = \frac{\lambda g(\omega)}{\alpha(\omega)},$$

and replacing it in Eq. (13), we see that

$$\begin{aligned} 2\pi i \ \mathrm{Res}\left[\frac{\lambda g(\omega)}{\alpha(\omega)} \ \tilde{\varphi}(\omega), z_0\right] &= 2\pi i \ \mathrm{Res}\left[\frac{\lambda^2 g^2(\tilde{z})}{\alpha^2(\tilde{z})}, \tilde{z} = z_0\right] \\ &= 2\pi i \ \mathrm{Res}\left[\frac{-i}{2\pi} \ \frac{1}{\alpha(\tilde{z})}, \tilde{z} = z_0\right] \\ &= -\frac{1}{\alpha'(z_0)} = -\frac{1}{\sqrt{\alpha'(z_0)}} \ \tilde{\varphi}_1, \end{aligned} \tag{15}$$

which is precisely that condition. This is also true for condition (14). Here we have used

$$\alpha_+(\omega) - \alpha_-(\omega) = -2\pi i \lambda^2 g^2(\omega).$$

On the other hand, one state whose initial values satisfy neither condition (13) nor condition (14) is, for example, the discrete eigenstate of the Hamiltonian,

$$\tilde{\varphi}_1=\tilde{\varphi}_1^\star=1 \quad \text{and} \quad \tilde{\varphi}(\omega)=\tilde{\varphi}^\star(\omega)=0,$$

that it is obviously out of $\mathscr{H}$.

Finally, we note that there are initial conditions which satisfy either (13) or (14) but not both, and other ones which satisfy neither, but are not eigenstates of H. Nevertheless, all of them obey the ''normalization'' condition (5).

IV. PROBABILITY BEHAVIOR OUT OF HILBERT

To characterize the physical behavior of non-$\mathscr{H}$ states, we should not consider the temporal evolution of the wave functions only, since each one alone does not provide information either about transition probabilities or about the mean values. For example, the discrete eigenstate $(\tilde{\varphi}_1,\tilde{\varphi}_1^\star)$ whose components evolve with e^{-iz_0t} and e^{iz_0t}, respectively, must not be interpreted as two decaying and growing states, because, in spite of the fact that they seem to be so, they are one single stationary state satisfying

$$\tilde{\varphi}_1(t)\tilde{\varphi}_1^\star(t)=\tilde{\varphi}_1(0)\tilde{\varphi}_1^\star(0)=1,$$

$$\tilde{\varphi}(\tilde{\omega},t)\tilde{\varphi}^\star(\tilde{\omega},t)=\tilde{\varphi}(\tilde{\omega},0)\tilde{\varphi}^\star(\tilde{\omega},0)=0,$$

as it may be seen from Eqs. (7)–(10).

Nevertheless, when evaluating probabilities for a general non-Hilbert state, including stationary ones, we see that—even though total ''probability'' is normalized to unity [see Eq. (5)]—the partial probability of finding the state in a bounded interval of energy may be out of the interval [0,1] or be a complex number. To investigate the nature of this problem let us consider an example. From Eqs. (7)–(10), the ''probability'' of being in the discrete eigenstate of H is

$$\varphi_1\varphi_1^\star=\frac{1}{\alpha'}, \tag{16}$$

and the ''probability density'' of finding the system with continuous energy ω is

$$\varphi(\omega)\varphi^\star(\omega)=\frac{1}{\alpha'}\,\frac{\lambda^2g^2(\omega)}{(z_0-\omega)^2}. \tag{17}$$

Expanding the probability of finding the system in the discrete state up to the second order in the interaction constant λ, we find

$$\varphi_1\varphi_1^\star=1-\lambda^2\int_0^\infty \frac{g^2(\omega)}{(\omega_0-\omega)^2}\,d\omega, \tag{18}$$

whereas for the probability density of finding it with a continuous energy ω, we obtain

$$\varphi(\omega)\varphi^\star(\omega)=\lambda^2\,\frac{g^2(\omega)}{(\omega_0-\omega)^2}. \tag{19}$$

So in this approximation we have

$$\varphi_1\varphi_1^\star+\int_0^\infty\varphi(\omega)\varphi^\star(\omega)d\omega=1.$$

We want to point out some features from the previous expressions

(1) Probability densities obey the condition expressed in (5).
(2) The probability density (19) has a pole in $\omega=\omega_0$.
(3) This pole produces complex ''probabilities'' in general and the ''probability density'' may be complex or greater than unity in its neighborhood.

Statements (1), (2), and (3) show us the nature of the problem and suggest a way to circumvent it. In fact, when we find ω_0 as the result of a measurement of the energy of the system, we do not know whether the corresponding state is the discrete eigenvector, the continuous vector with the same eigenvalue, or a linear combination of both. Then, it seems that we are not allowed to treat them as separated states assigning different probabilities to them because we cannot distinguish these states through energy measurements. But if we compute the probability to obtain ω_0 as the probability of the system to be in the discrete state plus the probability of the system to be in the continuum in a neighborhood of ω_0, the problem gets solved by itself because the infinite terms involving ω_0 cancel each other in the λ^2 approximation, i.e.,

$$\mathscr{P}[\omega_0-\Delta,\omega_0+\Delta]=\varphi_1\varphi_1^\star+\int_{\omega_0-\Delta}^{\omega_0+\Delta}\varphi(\omega_0)\varphi^\star(\omega_0)d\omega$$
$$=1-\lambda^2\int_0^{\omega_0-\Delta}\frac{g^2(\omega)}{(\omega_0-\omega)^2}\,d\omega-\lambda^2\int_{\omega_0+\Delta}^{\infty}\frac{g^2(\omega)}{(\omega_0-\omega)^2}\,d\omega,$$

does not have the problems mentioned in the above statements (2) and (3).

Nevertheless, this is only a second-order solution. It can be seen after a straightforward computation that in the next step of the expansion (fourth-order) complex contributions reappear from each term of the probability densities and do not cancel each other. This is so because, as we will see, this is not the right way to solve the problem.

Due to the bad behavior of probabilities out of $\mathscr{H}$, we have studied the behavior of another meaningful magnitude: the mean value. In the frame of the doublet representation we can generalize the expression of the free Hamiltonian mean value in any state as

$$\langle H_0(t)\rangle=\omega_0\varphi_1(t)\varphi_1^\star(t)+\int_0^\infty\omega\varphi(\omega,t)\varphi^\star(\omega,t)d\omega$$
$$=\omega_0\left[\frac{1}{\sqrt{\alpha'(z_0)}}\,\tilde{\varphi}_1e^{-iz_0t}+\int_0^\infty\frac{\lambda g(\tilde{\omega})}{\alpha_+(\tilde{\omega})}\,\tilde{\varphi}(\tilde{\omega})e^{-i\tilde{\omega}t}d\tilde{\omega}\right]$$
$$\cdot\left[\frac{1}{\sqrt{\alpha'(z_0)}}\,\tilde{\varphi}_1^\star e^{+iz_0t}+\int_0^\infty\frac{\lambda g(\tilde{\omega})}{\alpha_-(\tilde{\omega})}\,\tilde{\varphi}^\star(\tilde{\omega})e^{+i\tilde{\omega}t}d\tilde{\omega}\right]$$
$$+\int_0^\infty\left[\frac{1}{\sqrt{\alpha'(z_0)}}\,\frac{\lambda g(\omega)}{z_0-\omega}\,\tilde{\varphi}_1e^{-iz_0t}+\lambda^2g(\omega)\right.$$
$$\left.\times\int_0^\infty\frac{g(\tilde{\omega})}{\alpha_+(\tilde{\omega})(\tilde{\omega}-\omega)}\,\tilde{\varphi}(\tilde{\omega})e^{-i\tilde{\omega}t}d\tilde{\omega}+\tilde{\varphi}(\omega)e^{-i\omega t}\right]$$
$$\times\left[\frac{1}{\sqrt{\alpha'(z_0)}}\,\frac{\lambda g(\omega)}{z_0-\omega}\,\tilde{\varphi}_1^\star e^{+iz_0t}+\lambda^2g(\omega)\right.$$
$$\left.\times\int_0^\infty\frac{g(\tilde{\omega})}{\alpha_-(\tilde{\omega})(\tilde{\omega}-\omega)}\,\tilde{\varphi}^\star(\tilde{\omega})e^{+i\tilde{\omega}t}d\tilde{\omega}+\tilde{\varphi}^\star(\omega)e^{+i\omega t}\right]\omega d\omega,$$

whose instabilities come from the real contributions of exponentials $e^{\pm iz_0t}$. That is to say, the mean value has oscillating terms (which correspond to real frequencies ω) and growing or decaying waves coming from the imaginary contribution of the ''frequency'' z_0. These instabilities do not cause problems to Hilbert states because they satisfy conditions (13) and (14), so the complex exponentials cancel each other and do not produce vanishing or indefinitely growing terms. On the other hand, non-$\mathscr{H}$ states will have anomalous terms in the mean values of the Hamiltonian which diverge for growing time values, except when they are eigenstates of H. So here we also have a problem for the mean values evaluated in non-$\mathscr{H}$ states.

In conclusion, we point out that, in spite of the fact that it is possible to find the dynamical evolution for any doublet, it is difficult to assign probabilities or mean values (i.e., to give physical meaning) to them. In the next section we show that this problem is linked to the fact that the probability of measuring energy ω_0 has contributions coming from the discrete state plus contributions from any interval around ω_0, belonging to the continuum.

V. INTERACTIONS THAT GENERATE REAL z_0

In the previous sections we have identified the appearance of anomalous probabilities with z_0 being a nonreal number. Now we will show that the problems created by this complex quantity arise from the fact that the improper integrals around ω_0 yield a complex eigenvalue of the complete Hamiltonian. To show the explicit limit between complex and real eigenvalues, we make use of the solutions (7)–(10) which are continuous in λ (see Ref. 1). Thus, we can suppose that z_0 in expression (11) can be expanded as a power series of λ, taking $z_0|_{\lambda=0}=\omega_0$,

$$z_0=\omega_0+\lambda^2\int_0^\infty \frac{g^2(\omega)}{(\omega_0-\omega)}\,d\omega-\frac{2}{3}\lambda^4\int_0^\infty \frac{g^2(\omega)}{(\omega_0-\omega)}\,d\omega\int_0^\infty \frac{g^2(\omega')}{(\omega_0-\omega')^2}\,d\omega'+\cdots. \tag{20}$$

As we can see, improper integrals appear in the expansion of z_0. A generic coefficient in the series always contains a factor like

$$\frac{g^2(\omega)}{(\omega_0-\omega)^n},$$

implying the appearance of poles in the function under integration which—when computed in the complex plane—give imaginary contributions to z_0. These divergences point out the influence of the degeneration in the free Hamiltonian spectrum: when the eigenstate that corresponds to the discrete eigenvalue of the free Hamiltonian interacts with the eigenstate of the continuum with energy ω_0 and its neighborhood, complex eigenvalues of the complete Hamiltonian appear. As it has been shown in Ref. 2, the reality of the discrete eigenvalue z_0 is related to $g(\omega)$ evaluated in the eigenvalue.

Expression (20) tells us that, if we confine ourselves to the second-order expansion, z_0 is real when $g(\omega_0)=0$. This is in accordance with well-known results that establish that

$$g(\omega_0)=0\Rightarrow\Gamma^{(2)}=\pi\lambda^2 g^2(\omega_0)=0,$$

where $\Gamma^{(2)}$ is the second order in λ contribution to half the inverse of time life.[3–5] But this is not true for higher orders in the λ expansion because the condition $g^2(\omega_0)=0$ is not sufficient to guarantee that the whole integral be nondivergent.

From the previous results, we conclude that the condition that $g(\omega)$ must satisfy in order to obtain a real spectrum is

$$\lim_{\lambda\to 0}\frac{g^2(\omega)}{(z_0(\lambda)-\omega)^n}<\infty,\quad \forall n\in\mathcal{N}, \tag{21}$$

for any ω belonging to an interval that contains ω_0 because, when performing the expansion (20), condition (21) guarantees the reality of the expansion. This is equivalent to requesting that

$$\lim_{\lambda\to 0} z(\lambda)=\omega_0$$

does not belong to the support of $g(\omega)$.

A trivial example of interaction satisfying (21) is the Ohmic interaction $g(\omega)=\omega$ with a cutoff that leaves ω_0 out of the integration interval. In fact, any interaction that vanishes over a finite interval around ω_0 is a trivial example.

A nontrivial interaction satisfying (21) is

$$g(\omega)=\begin{cases}\exp(\omega_0-\omega)^{-1}, & \text{if } \omega_0<\omega;\\ 0, & \text{if } \omega_0=\omega;\\ \exp-(\omega_0-\omega)^{-1}, & \text{if } \omega_0>\omega,\end{cases} \tag{22}$$

which removes, in a continuous way, the interaction between the degenerate states of the free Hamiltonian.

For interactions satisfying condition (21), the spectrum of the complete Hamiltonian is real, no pathologies appear in the evaluation of probabilities, and the whole problem can be formulated in Hilbert space with no need of enlarging the state space to the doublet one in order to diagonalize H.

VI. A REDUCED SPACE

In general cases in which $g(\omega)$ does not satisfy condition (21) and z_0 is not necessarily a real number, we may remove the degeneration that causes anomalous probabilities defining a new basis with the following procedure (which we will develop in Dirac notation because it is clearer in this case).

To exclude the continuous eigenvalue ω_0 from the continuous spectrum (thus removing the degeneration) we define the set

$$\mathscr{C}=\{\omega/0\leqslant\omega\leqslant\omega_0-a \quad \text{or} \quad \omega_0+a\leqslant\omega\}, \tag{23}$$

with $a\in\mathfrak{R}$, $a>0$. Next, we define the orthogonal basis:

$$|\Lambda\rangle=\Lambda_1|1\rangle+\int_{\omega_0-a}^{\omega_0+a}\lambda(\omega)|\omega\rangle d\omega\,\epsilon\,\mathscr{H}, \tag{24}$$

$$\{|\omega_r\rangle\}=\{|\omega\rangle/\omega\in\mathscr{C}\}, \tag{25}$$

where the basis vectors satisfy

$$\langle\Lambda|\Lambda\rangle=\Lambda_1\Lambda_1^*+\int_{\omega_0-a}^{\omega_0+a}\Lambda(\omega)\Lambda^*(\omega)d\omega=1,$$
$$\langle\omega_r|\Lambda\rangle=\langle\Lambda|\omega_r\rangle=0 \quad \text{and} \quad \langle\omega_r|\omega_r'\rangle=\delta(\omega_r-\omega_r'). \tag{26}$$

In this basis the matrix elements of H are

$$\langle\Lambda|H|\Lambda\rangle=\omega_0\Lambda_1\Lambda_1^*+\lambda\int_{\omega_0-a}^{\omega_0+a}g(\omega)[\Lambda_1^*\Lambda(\omega)+\Lambda_1\Lambda^*(\omega)]d\omega$$
$$+\int_{\omega_0-a}^{\omega_0+a}\omega\Lambda(\omega)\Lambda^*(\omega)d\omega\equiv\Lambda(\lambda), \tag{27}$$

$$\langle\omega_r'|H|\omega_r\rangle=\omega_r\delta(\omega_r'-\omega_r), \tag{28}$$

$$\langle\omega_r|H|\Lambda\rangle=\lambda\Lambda_1 g(\omega_r). \tag{29}$$

With them, we can define the Hamiltonian $\tilde{H}$ which acts on the ‘‘reduced’’ space as

$$\tilde{H}=\Lambda(\lambda)|\Lambda\rangle\langle\Lambda|+\int_{\mathscr{C}}\omega|\omega\rangle\langle\omega|d\omega+\int_{\mathscr{C}}\lambda g(\omega)[\Lambda_1|\Lambda\rangle\langle\omega|+\Lambda_1^*|\omega\rangle\langle\Lambda|]d\omega. \tag{30}$$

Diagonalizing $\tilde{H}$, we obtain that its discrete eigenvalue satisfies the following recursive equation:

$$\tilde{\Lambda}(\lambda)=\Lambda(\lambda)+\lambda^2\int_{\mathscr{C}}\frac{\Lambda_1\Lambda_1^* g^2(\omega)}{(\tilde{\Lambda}(\lambda)-\omega)}\,d\omega. \tag{31}$$

Taking into account the results of Sec. V, we have that $\tilde{\Lambda}(\lambda)$ will be real (and the probability well defined) if $\lim_{\lambda\to 0}\tilde{\Lambda}(\lambda)$ does not belong to the support of $g(\omega)$, i.e., if

$$\lim_{\lambda\to 0}\tilde{\Lambda}(\lambda)=\lim_{\lambda\to 0}\Lambda(\lambda)=\omega_0+\int_{\omega_0-a}^{\omega_0+a}(\omega-\omega_0)\Lambda(\omega)\Lambda^*(\omega)d\omega \tag{32}$$

does not belong to the continuous set $\{\omega_r\}$ [we have used the normalization condition (26)]. As the function under integration in (32) is small enough near ω_0, it will be always possible to define some real number a near ω_0 to satisfy

$$\omega_0-a\leqslant\Lambda(\lambda=0)\leqslant\omega_0+a.$$

Then, we conclude that H has a real spectrum, the wave functions belong to $\mathscr{H}$, and the probabilities are well defined in the reduced space spanned by the basis (24)–(25).

We also notice that the reality of $\tilde{\Lambda}(\lambda)$ in (31) is guaranteed from a sufficiently small value of (ω_0-a) and it remains in the $a\to 0$ limit. In this case

$$\lim_{a\to 0}\tilde{\Lambda}\equiv W_0\in\mathfrak{R},$$

and the spectrum remains nondegenerated because the continuous ω_0 energy value has been removed from the continuous spectrum. So, the eigenfunction which evolves with $\exp(iz_0t)$ becomes the Hilbert eigenfunction which evolves with $\exp(iW_0t)$ defined in the reduced space.

VII. CONCLUSIONS

We have shown that the states which do not satisfy conditions (13) and (14) do not belong to $\mathscr{H}$. So, even though we can predict their temporal evolution, it is impossible to define either well-behaved probabilities or nondivergent mean values for them. Non-Hilbert states are useful to describe the dynamical evolution of physical wave functions, but they cannot be considered as having the same physical nature as those of Hilbert space (the same as plane waves do not have the same physical nature as ordinary Hilbert states).

Taking into account that the existence of non-Hilbert states is linked to the appearance of nonreal eigenvalues and that these eigenvalues are related to the degeneration of the Hamiltonian spectrum, we have shown how to solve the problem. To do so we define a reduced basis that generates wave functions which represent states with well-defined probabilities and mean values.

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