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Kaluza–Klein Cosmology with N Dilaton Fields

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The ground state for Kaluza–Klein cosmological models with more than one dilaton field is considered. The dimensional reduction is performed and the equations of motion for the dilaton fields are considered. The normal modes of oscillation are found, one of them, ψ, being the conformal factor in front of the metric for the true four-dimensional space-time. It is shown that a stable minimum exists when both the cosmological term and all the scalar curvatures of the extra-dimensional subspaces are negative. If all these scalar curvatures are positive, the extra-dimensional subspaces collapse and the quantum effects should be taken into account to stabilize them. All other combinations of the signs of scalar curvatures lead to decompactification of some of the subspaces. Some cosmological applications are discussed. One of them concerns the possibility of constructing Big-Bang cosmological models starting from a non-singular higher-dimensional space-time.

1. INTRODUCTION

Kaluza–Klein type theories became very popular during the last decade, because they provide quantum field theory with the desired symmetries in a natural geometrical way (see, for example, [1] and references cited therein). Moreover, the idea of extra dimensions, found some foundations in the modern string theory [2]. Applications to cosmology have also been considered ([3] and references cited therein), but in most of the cosmological applications, the ground state (only the conformal degree of freedom,

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all higher modes as well as gauge fields suppressed) contains only one dilaton field.

In this paper we consider a more general ground state, where the extra dimensions are divided into several compact subspaces, each of them endowed with a different space-time coordinate-dependent conformal factor. Taking a four-dimensional space-time, we thus obtain, in general, several dilaton fields coupled among themselves.

The aim of this work is to analyze the general conditions for the extra dimensions not decompactified, and discuss several possible cosmological implications.

In this work the potential $V(x)$ which determines the evolution of the system (compactification or decompactification) is not generated by external sources, but comes from the internal dimensions of the geometry. This approach allows us to study the qualitative behavior of the evolution of the universe in a non-*a-priori* fixed geometry. The type of evolution is determinated by the sign of the cosmological constant Λ and the sign of the curvature scalar of the internal subspaces. Particular vacuum solutions can be seen in [7], where the internal space geometry is an m-dimensional torus. External sources are studied, for instance in [8], where a perfect fluid in $(1+3+3)$-dimensions is studied. In [9], a 10-dimensional Robertson–Walker cosmological model is proposed, leading to the Higgs potential. Instead, we treat the problem in a less restrictive way (i.e., our generic n-dimensional geometry depends not only on the time variable, but on the four space-time coordinates).

The paper is organized as follows, in Section 2 the general formalism is developed, and the expression for the energy-momentum tensor for dilaton fields as well as the equations of motion are derived. The normal (physical) modes are found in Section 3. In Section 4 we consider two particular examples: namely, the cases of one and two dilaton fields. The condition for having a minimum in the potential is found, and also the condition for the extra dimensions to remain compactified. The general case is investigated in Section 5. The discussion completes the paper; there we consider some cosmological implications.

2. ENERGY MOMENTUM TENSOR AND FIELD EQUATIONS FOR DILATON FIELDS

We will consider the $d=m+n$ dimensional element

$$ds^2 = g_{AB}\, dz^A\, dz^B = g_{\mu\nu}\, dx^\mu\, dx^\nu + g_{ij}\, dy^i\, dy^j \tag{1}$$

where $A,\ B = 1,\dots,\ m+n;\ \mu,\ \nu = 1,\dots,\ m;\ i,\ j = 1,\dots,\ n$.

As we are interested in the ground state for a cosmological model, we have already suppressed the off-diagonal terms $g_{\mu j}$, and considered $g_{\mu\nu}$ as only a function of the space-time coordinates, i.e.,

$$g_{\mu\nu} = g_{\mu\nu}(x) \tag{2}$$

We use the following expression for the Ricci tensor

$$R_{AB} = \Gamma^{C}_{AB,C} - \Gamma^{C}_{AC,B} + \Gamma^{C}_{AB}\Gamma^{D}_{CD} - \Gamma^{C}_{AD}\Gamma^{D}_{BC} \tag{3}$$

where the Γ's are the Christoffel symbols. We choose the signature of the metric to be $(-, +, +, +, +, \ldots, +)$.

We will consider the g_{ij} metric as being

$$g_{i_l j_l}(x, y) = h^2_{(l)}(x)\, \tilde{g}_{i_l j_l}(y) \tag{4}$$

where $i_l, j_l, \ldots = 1, \ldots, n_{(l)}$; with $\sum n_{(l)} = n$, l: $1, \ldots, N$, and

$$\tilde{g}^{i_l k_l}\tilde{g}_{k_l j_l} = \delta^{i_l}_{j_l};\ \tilde{g}_{i_l j_l, k_p} = 0\ \forall p \neq l$$

Now the Ricci tensor for the complete metric reads

$$\begin{aligned}
R_{\mu\nu} &= \tilde{R}_{\mu\nu} - \sum_l \left(\frac{n_{(l)} h_{(l)\mu}}{h_{(l)}}\right)_{;\nu} - \sum_l \frac{n_{(l)} h_{(l)\mu} h_{(l)\nu}}{h^2_{(l)}} \\
R_{i_l\mu} &= 0 \\
R_{i_l k_l} &= \tilde{R}_{i_l k_l} - \tilde{g}_{i_l k_l}\left[\frac{h^{\lambda}_{(l);\lambda}}{h_{(l)}} - 2\frac{h^{\mu}_{(l)} h_{(l)\mu}}{h^2_{(l)}} + \frac{h^{\lambda}_{(l)}}{h_{(l)}} \sum_{l'} \frac{n_{(l')} h_{(l')\lambda}}{h_{(l')}}\right]
\end{aligned} \tag{5}$$

where $\tilde{R}_{\mu\nu}$ is built from $\Gamma^{\lambda}_{\mu\nu}(x)$ only and $\tilde{R}_{i_l k_l}$ is built from the $\Gamma^{k}_{ij}(y)$. Here ; means covariant derivative with respect to the metric $g_{\mu\nu}$, and $h_{(l)\lambda}$ means $\partial_\lambda h_{(l)}$.

Writing the total action

$$S = \frac{1}{16\pi G_{\mathscr{D}}} \int \sqrt{|{}^{\mathscr{D}}g|} \cdot (R - 2\Lambda)\, d^{\mathscr{D}}z; \quad R = g^{AB} R_{AB}$$

splitting it into space-time and extra dimensions parts, and then integrating over the extra dimensions variables we obtain

$$S = \frac{\prod_l V_{(l)}}{16\pi G_{\mathscr{D}}} \int \prod_l h^{n_l}_{(l)} \sqrt{|{}^m g|}\ ({}^m R - 2\Lambda + \text{other terms})\, d^m x$$

where ${}^m g$ and ${}^m R$ are the metric and scalar curvature of the m-dimensional space-time respectively.

In order to obtain the usual Einstein action, we perform the following transformation

$$g_{\mu\nu}=\psi^2\gamma_{\mu\nu} \qquad \text{with} \qquad \psi^2=\left(\prod_l h_{(l)}^{n_l}\right)^{-(2)/(m-2)} \tag{6}$$

Taking into account Eqs. (5) and (6) we write the Einstein equations

$$G_{AB}=R_{AB}-\tfrac{1}{2}Rg_{AB}+\Lambda g_{AB}=0$$

as

$$\begin{aligned} G_{\mu\nu}=\tilde{\tilde{G}}_{\mu\nu}-\frac{\psi^2}{2}\gamma_{\mu\nu}\sum_l \frac{\tilde{R}_{(l)}}{h_{(l)}^2}-\frac{1}{m-2}\sum_{ll'}\frac{n_l n_{l'} h_{(l)\mu}h_{(l')\nu}}{h_{(l)}h_{(l')}}-\sum_l n_l \frac{h_{(l)\mu}h_{(l)\nu}}{h_{(l)}^2} \\ +\frac{1}{2}\gamma_{\mu\nu}\left[\frac{1}{m-2}\sum_{ll'}\frac{n_l n_{l'} h^{\lambda}_{(l)}h_{(l')\lambda}}{h_{(l)}h_{(l')}}+\sum_l n_l \frac{h^{\lambda}_{(l)}h_{(l)\lambda}}{h_{(l)}^2}\right]=-\Lambda\psi^2\gamma_{\mu\nu} \end{aligned} \tag{7}$$

$$\begin{aligned} G_{i_lk_l}=\tilde{G}_{i_lk_l}-\frac{h_{(l)}^2}{\psi^2}\tilde{g}_{i_lk_l}\left(\frac{h^{\lambda}_{(l)}}{h_{(l)}}\right)_{|\lambda}-\frac{h_{(l)}^2}{2\psi^2}\tilde{g}_{i_lk_l}\tilde{\tilde{R}} \\ -\frac{h_{(l)}^2}{2}\tilde{g}_{i_lk_l}\sum_{l\neq l'}\frac{\tilde{R}_{(l')}}{h_{(l')}^2}+\frac{h_{(l)}^2}{2\psi^2}\tilde{g}_{i_lk_l}\left\{\frac{1}{m-2}\sum_{ll'}n_l n_{l'}\frac{h^{\lambda}_{(l)}h_{(l')\lambda}}{h_{(l)}h_{(l')}}\right. \\ \left.+\sum_{l'}n_{l'}\left[\frac{h^{\lambda}_{(l')}h_{(l')\lambda}}{h_{(l')}^2}-\frac{2}{m-2}\left(\frac{h^{\lambda}_{(l')}}{h_{(l')}}\right)_{|\lambda}\right]\right\} \\ =-\Lambda h_{(l)}^2\tilde{g}_{i_lk_l} \end{aligned} \tag{8}$$

$\tilde{\tilde{G}}_{\mu\nu}$ is built up with $\gamma_{\mu\nu}$ and $|$ is the covariant derivative with respect to $\gamma_{\mu\nu}$. Here ψ_μ means $\partial_\mu\psi$.

We see from Eq. (8) that $\tilde{G}_{i_lk_l}$ is proportional to $\tilde{g}_{i_lk_l}$. Using this fact and $\tilde{G}_{i_lk_l;}{}^{kl}=0$, we obtain that the factor is constant, so every $\tilde{R}^{i_l}_{i_l}=\tilde{R}_l$ is also constant.

Defining $\varphi_{(l)}$ through $h_{(l)}=\ln(\varphi_{(l)})$, we are able to write Eq. (7) as

$$\tilde{\tilde{G}}_{\mu\nu}=8\pi G T_{\mu\nu} \tag{9}$$

where

$$\begin{aligned} 8\pi G T_{\mu\nu}=\frac{1}{m-2}\left[\sum_{ll'}n_{(l)}n_{(l')}\varphi_{(l)\mu}\varphi_{(l')\nu}+(m-2)\sum_l n_l\varphi_{(l)\mu}\varphi_{(l)\nu}\right] \\ -\gamma_{\mu\nu}\left\{\frac{1}{2(m-2)}\left[\sum_{ll'}n_l n_{l'}\varphi^{\lambda}_{(l)}\varphi_{(l')\lambda}+(m-2)\sum_l n_l\varphi^{\lambda}_{(l)}\varphi_{(l)\lambda}\right]+V\right\} \end{aligned} \tag{10}$$

V being a potential, given by

$$V=\tfrac{1}{2}e^{-(1)/(m-2)\sum_l n_l\varphi_{(l)}}\left[2\Lambda-\sum_l \tilde{R}_{(l)}e^{-2\varphi_{(l)}}\right] \tag{11}$$

Where by we see that Eq. (9) represents the Einstein equations for the metric $\gamma_{\mu\nu}$ with a set of scalar fields acting as sources.

We recall that the $\varphi_{(l)}$ are not uniquely defined; because $h_{(l)}$, are defined up to a constant factor.

Using Eq. (8), we find the field equations for the dilaton fields

$$\begin{aligned}\Box\varphi_{(l)}=\frac{1}{m-2+n_l}\Bigg\{&\sum_{l'} e^{-(2)/(m-2)\sum_{l''} n_{l''}\varphi_{(l'')}}\tilde{R}_{(l')}\\&+\frac{m-2}{n_l}e^{-(2)/(m-2)\sum_{l'} n_{l'}\varphi_{(l')}}\tilde{R}_{(l)}\\&-2\Lambda e^{-(2)/(m-2)\sum_{l'} n_{l'}\varphi_{(l')}}-\sum_{l\neq l'} n_{l'}\Box\varphi_{(l')}\Bigg\}\end{aligned} \tag{12}$$

It is easy to see that the same equations can be obtained from the conservation of the energy-momentum tensor, i.e., $T^{\nu}_{\mu|\nu}=0$.

From the energy-momentum tensor (10) we can immediately write down the Lagrangian

$$\mathcal{L}=T-V=\frac{1}{2(m-2)}\sum_{ll'} n_{l'}n_{l'}\varphi^{\lambda}_{(l)}\varphi_{(l')\lambda}-\frac{1}{2}\sum_l n_l\varphi^{\lambda}_{(l)}\varphi_{(l)\lambda}-V$$

where

$$T=-\frac{1}{2}\left\{\frac{1}{m-2}\sum_{ll'} n_l n_{l'}\varphi^{\lambda}_{(l)}\varphi_{(l')\lambda}+\sum_l n_l\varphi^{\lambda}_{(l)}\varphi_{(l)\lambda}\right\}$$

is the kinetic energy of the dilaton fields. We see that all the dilaton fields are mixed; thus, to define the physical fields we must find the transformation to the normal modes.

3. NORMAL MODES FOR THE SCALAR DILATON FIELDS

To obtain the normal modes of the dilaton fields we diagonalize the expression for the kinetic energy; which can be written in a matrix form as follows

$$T=\frac{1}{2}\bar{\nabla}\bar{\varphi}^T\bar{\Pi}\bar{\nabla}\bar{\varphi}$$

$$=\frac{1}{2}\bar{\nabla}\bar{\varphi}^T\left\{\frac{1}{m-2}\begin{pmatrix}1 & 1 & \cdots & 1\\ 1 & & & \\ \vdots & & & \\ 1 & \cdots & \cdots & 1\end{pmatrix}+\text{diag}\,(1,\ldots,1)\right\}\bar{\nabla}\bar{\varphi} \qquad (13)$$

where we have taken

$$\bar{\varphi}=\begin{bmatrix}\varphi_1\\ \varphi_1\\ \vdots\\ \varphi_1\\ \varphi_2\\ \vdots\\ \varphi_2\\ \vdots\\ \varphi_k\\ \vdots\\ \varphi_k\end{bmatrix}\begin{matrix}\left.\begin{matrix}\\ \\ \\ \\ \end{matrix}\right\} n_1\\ \left.\begin{matrix}\\ \\ \\ \end{matrix}\right\} n_2\\ \\ \left.\begin{matrix}\\ \\ \\ \end{matrix}\right\} n_k\end{matrix}$$

The next step is to find a matrix C (with elements C_{ij}) which makes the following transformations

$$\begin{cases}\bar{\Pi}=C^T\tilde{\bar{\Pi}}C\\ \tilde{\varphi}=C\varphi\end{cases}\text{with } \tilde{\bar{\Pi}}=\text{diag}(\lambda_1,\ldots,\lambda_n),$$

where λ_m are the eigenvalues of $\bar{\Pi}$ and $C^T=C^{-1}$.

Let us now consider the form of the following matricial equation $C\bar{\Pi}=\tilde{\bar{T}}C$, i.e.,

$$C\bar{\Pi}=\frac{1}{m-2}\begin{pmatrix}C_1 & \cdots & C_1\\ C_2 & \cdots & C_2\\ \vdots & & \vdots\\ C_n & & C_n\end{pmatrix}+C$$

$$=\tilde{\bar{\Pi}}C$$

$$=\begin{pmatrix}\lambda_1 C_{11} & \lambda_1 C_{12} & \cdots & \lambda_1 C_{1n}\\ \lambda_2 C_{21} & & & \lambda_2 C_{2n}\\ \vdots & & & \\ \lambda_n C_{n1} & & & \lambda_n C_{nn}\end{pmatrix} \qquad (14)$$

with $C_j=\sum_{k=1}^{n} C_{jk}$

Equation (14) implies

$$\frac{C_k}{m-2} = (\lambda_k - 1)\, C_{kl} \qquad \forall k, l \tag{15}$$

Summing over l we obtain

$$\frac{nC_k}{m-2} = (\lambda_k - 1)\, C_k \tag{16}$$

Two cases can be considered

i. $C_k = 0$. If $\lambda_k = 1$, C would have no inverse matrix, so we would not be able to find a transformation between φ and $\tilde{\varphi}$. We conclude, then, that we need to have $\lambda_k \neq 1$.

ii. $C_k \neq 0$. Then we obtain $\lambda_k = (m+n-2)/(m-2)$ and some of them may be $\lambda_k = 1$. Taking the trace of $\bar{\Pi} = C^T \tilde{\bar{\Pi}} C$, and taking into account that C is an orthonormal matrix, it is easy to show that the eigenvalue $\lambda_1 = (m+n-2)/(m-2)$ appears only once, while $\lambda = 1$ is $(n-1)$ times degenerated. We will choose λ_1 to be the first eigenvalue.

Using the orthogonality condition, we obtain the following general form for the matrix C:

$$C = \begin{pmatrix} 1/\sqrt{n} & \cdots & 1/\sqrt{n} \\ C_{21} & & C_{2n} \\ \vdots & & \vdots \\ C_{n1} & \cdots & C_{nm} \end{pmatrix} \tag{17}$$

where its coefficients obey the following constraints

$$\begin{aligned} \sum_l C_{jl} &= 0 \qquad \forall j \geqslant 2 \\ \sum_l C_{jl} C_{kl} &= \delta_{jk} \end{aligned} \tag{18}$$

Written as a function of the new fields the potential V has the following structure

$$\begin{aligned} V = e^{-(\sqrt{n})/(m-2)\,\tilde{\varphi}_1} \{ \Lambda - e^{-(2)/(\sqrt{n})\,\tilde{\varphi}_1} [\tilde{R}_1 e^{-(2)/(\sqrt{n})\, f_1(\tilde{\varphi}_2, \ldots, \tilde{\varphi}_l)} \\ + \tilde{R}_2 e^{-(2)/(\sqrt{n})\, f_2(\tilde{\varphi}_2, \ldots, \tilde{\varphi}_l)} + \cdots + \tilde{R}_n e^{-(2)/(\sqrt{n})\, f_n(\tilde{\varphi}_2, \ldots, \tilde{\varphi}_l)}] \} \end{aligned} \tag{19}$$

After the diagonalization, it is easy to write the Euler–Lagrange equations for the normal modes. We get

$$\tilde{\varphi}^{\lambda}_{(l)|\lambda} = -\frac{\partial V}{\partial \tilde{\varphi}_{(l)}}$$

Note that they are just the transformed equations for the geometrical dilaton fields $\varphi_{(l)}$ [Eq. (12)], which we have obtained from the higher-dimensional Einstein equations.

Thus we have proved the existence of the normal modes, and are now able to apply the well-known theorems of ordinary mechanics in order to understand the evolution of the system under consideration. However, first we will consider two illustrative examples.

4. EXAMPLES

4.1. One Dilaton Field

Einstein equations for one dilaton field now read

$$\begin{aligned} G_{\mu\nu} &= \tilde{\tilde{G}}_{\mu\nu} - \frac{n(m+n-2)}{m-2}\varphi_{\mu}\varphi_{\nu} + \frac{n(m+n-2)}{2(m-2)}\varphi_{\lambda}\varphi^{\lambda}\gamma_{\mu\nu} \\ &\quad + \frac{1}{2}e^{(2(m+n-2)/(m-2))\varphi}\tilde{R}\gamma_{\mu\nu} = \Lambda e^{-(2n/(m-2))\varphi}\gamma_{\mu\nu} \\ G_{ik} &= \tilde{G}_{ik} - \frac{m+n-2}{2(m-2)}e^{(2(m+n-2)/(m-2))\varphi}\tilde{g}_{ik}(2\varphi^{\lambda}_{|\lambda} - n\varphi_{\lambda}\varphi^{\lambda}) \\ &\quad + \frac{1}{2}e^{(2(m+n-2)/(m-2))\varphi}\tilde{g}_{ik}\tilde{\tilde{R}} = \Lambda e^{2\varphi}\tilde{g}_{ik} \end{aligned} \tag{20}$$

where quantities with $\approx$ are constructed from $\gamma_{\mu\nu}$ and those with $\sim$ from $\tilde{g}_{ik}$.

The additional terms appearing in the first equation can be understood as the energy momentum tensor of the dilaton field

$$\begin{aligned} T_{\mu\nu} &= \frac{n(m+n-2)}{m-2}\left[\varphi_{\mu}\varphi_{\nu} - \frac{1}{2}\varphi_{\lambda}\varphi^{\lambda}\gamma_{\mu\nu}\right] \\ &\quad + \left[-\frac{1}{2}e^{-(2(m+n-2)/(m-2))\varphi}\tilde{R} + \Lambda e^{-(2n/(m-2))\varphi}\right]\gamma_{\mu\nu} \end{aligned} \tag{21}$$

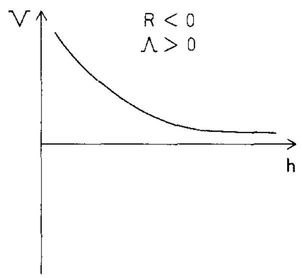

Fig. 1. The effective potential V as a function of the metric factor h for a positive value of the cosmological constant Λ and a negative value of the curvature $\tilde{R}$ of the extra dimensions. In this case for any given initial conditions, the system tends to evolve towards larger values of h (decompactification).

Where by we can read off the effective potential

$$V(\varphi) = -\tfrac{1}{2}e^{-(2(m+n-2)/(m-2))\varphi}\tilde{R} + \Lambda e^{-(2n/(m-2))\varphi} \tag{22}$$

Now, we are able to study the effective potential as a function of two parameters, i.e., Λ and $\tilde{R}$.

Four different cases appear, depending on the sign of Λ and $\tilde{R}$. They are shown in Figs. 1–4. One can easily see that cases 1 and 4 represent decompactification and compactification (with $V<0$), respectively; that is given a value of the kinetic energy of the field, it will evolve to infinitely large values in case 1, and to zero in case 4.

Cases 2 and 3 are richer, because they show an extremum in the effective potential. In case 2 one would have a stable point at the minimum of the potential ($V_{\min}<0$), while in case 3 there is a maximum and it will depend on the initial conditions of the field whether we will have a case of compactification or decompactification.

Now, we will compute the value of the potential at the extremum.

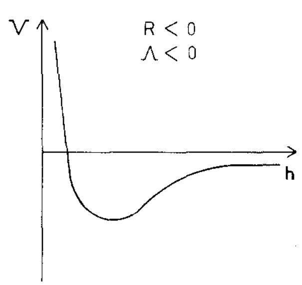

Fig. 2. The effective potential V as a function of the metric factor h for $\Lambda<0$ and $\tilde{R}<0$. In this case V has a minimum around which the system is able to stabilize by itself.

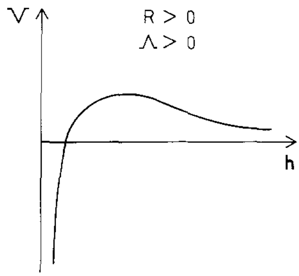

Fig. 3. The effective potential V as a function of h shows a cusp for $\Lambda>0$ and $\tilde{R}>0$. Depending on the initial value of h (whether h_{initial} is bigger or smaller than $h_{\text{cusp.}}$), the system will evolve to decompactification or compactification.

Going back to $h=\exp(\varphi)$ the effective potential will be

$$V=h^{-(2n/(m-2))}\left(-\tfrac{1}{2}h^{-2}\tilde{R}+\Lambda\right) \tag{23}$$

Differentiation gives the condition for extrema

$$\frac{\partial V}{\partial h}=0 \Rightarrow h^2_{\text{extr}}=\frac{m+n-2}{n}\frac{\tilde{R}}{\Lambda} \tag{24}$$

We see that the condition for having extrema is fulfilled when $\tilde{R}$ and Λ have the same sign. So,

$$V_{\text{extr}}=\frac{\Lambda}{2}\frac{(2m+n-2)}{m+n-2}\left[\frac{m+n-2}{n}\frac{\tilde{R}}{\Lambda}\right]^{-(n/(m-2))} \tag{25}$$

We see that for negative values of Λ we will obtain a minimum, where $V_{\min}$ is negative (Fig. 2), while for positive values of Λ we will obtain a maximum, where $V_{\max}$ is positive (Fig. 3).

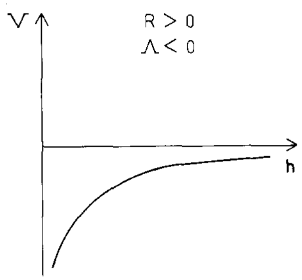

Fig. 4. When $\Lambda<0$ and $\tilde{R}>0$, the form of the effective potential is such that for any initial value of h the dilaton fields evolve to a compactified value.

For the sake of completeness we write down the equation of motion for φ. This can be deduced from the expression (21) for the $T_{\mu\nu}$ and then, computing $T^{\nu}_{\mu|\nu}=0$ or working out Eq. (20) to eliminate $\tilde{\tilde{R}}$

$$\varphi^{\lambda}_{|\lambda}=\frac{2}{m+n-2}\Lambda e^{-(2n/(m-2))\varphi} - \frac{\tilde{R}}{n}e^{-(2(m+n-2)/(m-2))\varphi}=-\frac{\partial V}{\partial\varphi} \tag{26}$$

4.2. Two Dilaton Fields

Let us now consider the case of two dilaton fields, one of them in k dimensions and the other in $n-k$ dimensions. They are represented by the vector

$$\bar{\varphi}=\begin{pmatrix}\varphi_1\\ \vdots\\ \varphi_1\\ \varphi_2\\ \vdots\\ \varphi_2\end{pmatrix}\begin{matrix}\left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\}k\\ \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\}n-k\end{matrix} \tag{27}$$

From the general expression (11) for the effective potential we obtain

$$V=\tfrac{1}{2}e^{-(1/(m-2))[k\varphi_1+(n-k)\varphi_2]}\{2\Lambda-\tilde{R}_1e^{-2\varphi_1}-\tilde{R}_2e^{-2\varphi_2}\} \tag{28}$$

In Section 3 we have shown that we must "rotate" the system of fields φ_1, φ_2 to the normal modes $\tilde{\varphi}_1$, $\tilde{\varphi}_2$ to diagonalize the kinetic energy. To do so, we perform the following transformation [after using matrix C, Eq. (5)]

$$\begin{aligned}\varphi_1&=\frac{1}{\sqrt{n}}\tilde{\varphi}_1+\frac{n-k}{nC_2}\tilde{\varphi}_2\\ \varphi_2&=\frac{1}{\sqrt{n}}\tilde{\varphi}_1-\frac{k}{nC_2}\tilde{\varphi}_2\end{aligned} \tag{29}$$

where

$$C_2=\sum_{j=1}^{k}C_{2j}$$

Now, we are able to express the effective potential in terms of the normal modes

$$V=\tfrac{1}{2}e^{-(\sqrt{n}/(m-2))\tilde{\varphi}_1}\left[2\Lambda-e^{-(2\tilde{\varphi}_1/\sqrt{n})}\left(\tilde{R}_1 e^{-(2/C_2)\varphi_2(1-(k/n))}\right.\right.$$
$$\left.\left.+\tilde{R}_2 e^{(2/C_2)(k/n)\tilde{\varphi}_2}\right)\right] \tag{30}$$

Considering the condition for extrema in $\tilde{\varphi}_2$, keeping $\tilde{\varphi}_1$ constant, i.e.,

$$\left.\frac{\partial V}{\partial \tilde{\varphi}_2}\right|_{\tilde{\varphi}_1=\text{const}}=0$$

we obtain

$$e^{(2/C_2)\tilde{\varphi}_2^{(\text{extr})}}=\frac{\tilde{R}_1}{\tilde{R}_2}\frac{n-k}{k} \tag{31}$$

This equation shows that to have extrema, $\tilde{R}_1$ and $\tilde{R}_2$ must have the same sign.

The condition for a minimum

$$\left.\frac{\partial^2 V}{\partial \tilde{\varphi}_2^2}\right|_{\varphi_2^{\min}}>0$$

gives

$$\tilde{R}_1<0 \qquad \text{and} \qquad \tilde{R}_2<0 \tag{32}$$

Studying the conditions to have extrema in $\tilde{\varphi}_1$ we get

$$e^{-(\sqrt{n}/(m-2))\tilde{\varphi}_1^{(\text{extr})}}=\frac{\Lambda\mathscr{D}^{-1}n}{2(m-2)+n} \tag{33}$$

where

$$\mathscr{D}=\tfrac{1}{2}\left[\tilde{R}_1 e^{-(2/C_2)\varphi_1(1-(k/n))}+\tilde{R}_2 e^{(2/C_2)(k/n)\tilde{\varphi}_2}\right]$$

We see that Λ and $\mathscr{D}$ must have the same sign.

When we impose the condition for having a minimum on the second derivative of V, we obtain

$$\Lambda<0 \tag{34}$$

So, conditions (32) and (34) must be satisfied to have a minimum in both variables, while all the inequalities must be reversed to have a maximum.

The other possibilities of combinations of signs can be studied straightforwardly from the expression (30) for the potential. The result is

$$\Lambda > 0, \quad \tilde{R}_1 < 0, \quad \tilde{R}_2 < 0 \qquad \text{decompactification}$$

$$\Lambda < 0, \quad \tilde{R}_1 > 0, \quad \tilde{R}_2 > 0 \qquad \text{compactification}$$

and the remaining combinations, where just one $\tilde{R}_{(l)}$ is smaller than zero, gives decompactification in this coordinate $h_{(l)}$.

This picture for the two dilaton fields resembles that one about the previous example. So, one can imagine for this second example that Figs. 1–4 have an extra axis but with the same qualitative behavior for the effective potential.

The results of these two examples lead us to the generalization of the following section.

5. GENERAL CONDITIONS FOR COMPACTIFICATION AND MINIMUM IN V

Now that we have illustrated some aspects of the dilaton fields, we will show on general grounds that this behavior is not a particular characteristic of the examples considered in the last section, but a general qualitative behavior.

Given an initial value of the $\varphi_{(l)}$, the system of fields will evolve to lower values of the effective potential, following the steepest direction of variation, given by $\bar{\nabla} V$, the gradient of V.

We are mainly interested in studying the phenomenon of compactification (i.e., $h_{(l)} \to 0$, so $\varphi_{(l)} \to -\infty$). Then, the mathematical condition to impose to the potential is that its gradient must lie in the first quadrant formed when one plots V as a function of the variables $\varphi_{(l)}$. That is,

$$\frac{\partial V}{\partial \varphi_i} > 0 \tag{35}$$

Taking the first partial derivative of the potential given by (11), we obtain

$$\frac{\partial V}{\partial \varphi_i} = -2e^{-2\sum_j (n_j \varphi_j/(m-2))} \left(\frac{n_i}{m-2} \Lambda - \tilde{R}_{(i)} e^{-2\varphi_i} - \frac{n_i}{m-2} \sum_l \tilde{R}_{(l)} e^{-2\varphi_l} \right) \tag{36}$$

Then, the condition for compactification (35) will be

$$\Lambda - \frac{m-2}{n_i} \tilde{R}_{(i)} e^{-2\varphi(i)} - \sum_l \tilde{R}_{(l)} e^{-2\varphi(l)} < 0 \tag{37}$$

Now, let us suppose that any $\tilde{R}_{(j)} > 0$. Then for Λ and $\varphi_{(l)}(l \neq j)$ fixed, the last inequality will not be fulfilled when $\varphi_{(j)} \to -\infty$. So, to have compactification we must have

$$\tilde{R}_{(l)} > 0 \qquad \forall l, \text{ for any sign of } \Lambda \tag{38}$$

We will turn now to study the conditions for V to have a minimum, i.e.,

$$\left.\frac{\partial V}{\partial \varphi_i}\right|_{\varphi = \varphi_{\min}} = 0 \qquad \text{and} \qquad \left.\frac{\partial^2 V}{\partial \varphi_i^2}\right|_{\varphi = \varphi_{\min}} > 0 \tag{39}$$

From Eq. (35)

$$\Lambda - \sum_l \tilde{R}_{(l)} e^{-2\varphi(l)} - \frac{m-2}{n_i} \tilde{R}_{(i)} e^{-2\varphi(i)} = 0 \tag{40}$$

multiplying by n_i and summing we get

$$n\Lambda - (n+m-2) \sum_l \tilde{R}_{(l)} e^{-2\varphi(l)} = 0 \tag{41}$$

So, replacing this expression for $\sum_l \tilde{R}_{(l)} e^{-2\varphi(l)}$ into (40) we obtain

$$\tilde{R}_i e^{-2\varphi(i)\min} = \frac{n_i \Lambda}{m-2+n} \tag{42}$$

This is, in fact, the condition for extremum; to be a minimum the second condition in (39) must be satisfied

$$\frac{\partial^2 V}{\partial \varphi_{(i)}^2} = \left(\frac{2n_i}{m-2}\right)^2 e^{-2\sum_j (n_j \varphi_{(j)}/(m-2))} \left[\Lambda - \sum_l \tilde{R}_{(l)} e^{-2\varphi(l)} - \frac{m-2}{n_i}\left(2 + \frac{m-2}{n_i}\right) \tilde{R}_{(i)} e^{-2\varphi(i)}\right] \tag{43}$$

Computing its value at the minimum with the aid of Eq. (42)

$$\left.\frac{\partial^2 V}{\partial \varphi_{(i)}^2}\right|_{\min} = \left(\frac{2n_i}{m-2}\right)^2 e^{-2\sum_j (n_j \varphi_{(j)}/(m-2))} \times \left[-\Lambda \frac{(m-2)\left(1 + \frac{m-2}{n_i}\right)}{m+n-2}\right] > 0 \tag{44}$$

So, for $m > 2$ (the space-times we are interested in) Λ must be negative and, from Eq. (42), all $\tilde{R}_{(l)}$ must be negative in order to have a minimum in V.

$$\Lambda < 0 \qquad \text{and} \qquad \tilde{R}_{(l)} < 0 \ \forall l \rightarrow \text{minimum in } V \tag{45}$$

For the sake of completeness we give the value of the potential at the minimum

$$V_{\min} = \Lambda \left(\frac{m-2}{m-2+n}\right) \prod_i \left(\frac{n_i \Lambda \tilde{R}_{(i)}^{-1}}{m-2+n}\right)^{n_i/(m-2)} \tag{46}$$

6. DISCUSSION

We have stressed in Section 4 that when $\Lambda < 0$ and $\tilde{R}_{(l)} < 0$ the effective potential has a minimum that occurs at negative values. When dealing with gravitation, energy must be positive. So, we have to consider oscillations around the minimum of the potential, in such a way that each field carries with it a kinetic energy large enough to reach a positive total energy (in spite of having negative energy for each individual field).

Also, due to gravitational friction, the total energy density is not conserved; it diminishes during the expansion of the universe and increases during the contraction. To maintain the positive energy density we need to have oscillation of (some) normal modes. It would be interesting to take these facts into account in the inflationary scenario because inflation reduces the fluctuations.

The results we have obtained let us speculate about the possibility of constructing Big-Bang cosmological models starting from a nonsingular multidimensional space-time [one of the h tends to infinity while the corresponding scalar factor of the cosmological model tends to zero, see Eq. (6)].

We saw that the input parameters of the effective potential are the cosmological Λ term, and the constant scalar curvatures of the extra-dimensional subspaces, $\tilde{R}_{(l)}$. The qualitative behavior of the system of coupled scalar fields depends on the signs of these constant curvatures. It should be noted that the sign of a scalar curvature depends also on the signature of the subspace metric. For example, in order to obtain a negative value for a scalar curvature we can use either a negative curvature submanifold if the metric signature is positive (space-like coordinate in this submanifolds) or some positive curvature submanifold if the metric is negative (time-like coordinates). This possibility was first considered in [4].

It is possible, in principle, to consider the phase transitions from one pattern of subspaces to another pattern differing, say, by the number of dilaton fields or even by the sign of the scalar curvature. For this purpose, the formalism of the thin shells, first developed in [5] and applied to cosmological problems in [6], can be used. However, contrary to the ordinary four-dimensional cosmology, here we would need to consider a two-bubble-wall system in order to keep the dilaton fields and, thus, the extra-dimensional metric coefficients continuous across the walls. Then, these phases (patterns of subspaces) will survive as the volume increases. In this way the additional constraints on the structure of the extra-dimensional space can be obtained. However, such an investigation requires the knowledge of the solution between two bubble walls; this solution would be necessarily different from the ground state considered here (massive modes should be taken into account). We will investigate this problem in a subsequent paper.

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