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GRAVITATIONAL AND MATTER ENERGY–MOMENTUM DENSITIES AND EQUIVALENCE PRINCIPLE IN NON-RIEMANNIAN GEOMETRIES

M. CASTAGNINO*

Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Argentina

and

Instituto de Astronomía y Física del Espacio, C.C. 67 Suc 28, 1428 Buenos Aires, Argentina

M. L. LEVINAS* and N. UMÉREZ*

Instituto de Astronomía y Física del Espacio, C.C. 67 Suc 28, 1428 Buenos Aires, Argentina

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We introduce an energy–momentum density vector which is independent of the affine structure of the manifold. Conservation of this quantity is linked to observers. Integrating over timelike surfaces, we define the Hamiltonian and momentum of the system which coincide with the corresponding standard ADM definitions when taking adequate asymptotical conditions. We define an Equivalence Principle for manifolds with torsion as a possible extension of the Equivalence Principle of General Relativity to non-Riemannian geometries.

1. Introduction

If we want to extend The Equivalence Principle of General Relativity (GR) to general metric manifolds (eventually with torsion), it is necessary to study the local behavior of energy and momentum and introduce local systems of reference that will play the role of free falling observers. So we need to introduce the notion of observers in arbitrary curved space–time. Their trajectories will be the result of a generic diffeomorphism over the given manifold. In principle, observers may follow any trajectories, not necessarily free falling ones. As in Minkowski space–time where we can deal with inertial or noninertial observers — e.g. Rindler observers — in curved space–time we can deal with geodesic or general and arbitrary non-geodesic observers; but for free falling observers, energy and momentum will be

*Fellow of the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina.

locally conserved. We show that these conditions are satisfied if the Lie derivative of the tetrads, defining the free falling systems, locally vanish. Nevertheless, this condition does not imply the local annihilation of the whole non-Riemannian connection.

For our treatment, we need to introduce a definition of energy and momentum applicable to manifolds with torsion. We discuss this point briefly.

In Classical Mechanics as well as in Special Relativity (SR), energy and momentum can be well defined because the manifolds where these theories are formulated admit global symmetries: invariances under Galileo and global Poincaré transformations respectively. This is not the case of curved manifolds where global space–time symmetry is a meaningless concept. Energy and momentum in general, curved space–time are directly related to the local Poincaré group. This group plays the role of the global Poincaré group in flat space–time manifolds and it is its natural extension.[1] But in general, in addition to local symmetry, some geometrical restrictions should be imposed: for example, conserved quantities are introduced imposing asymptotically flat spaces[2,3] or constant curvature.[4,5] It is also possible to obtain a covariant Hamiltonian including a covariant expression for the conserved quantities of an asymptotically flat or constant curvature space.[6] More recently, in Ref. 7, it has been shown that in order to define energy, the same surface term may be used in both, the Hamiltonian and the action.

Another way to define energy and momentum is to generalize the notion of energy–momentum tensor from SR to curved space–time replacing ordinary derivatives by covariant derivatives in its definition (minimal coupling in the Lagrangian). Then the local concept of energy–momentum density in curve space–time is transformed into a global definition of energy and momentum integrating over timelike hypersurfaces of the space–time manifold. But when this procedure is applied to theories with torsion, a problem appears: flatness with torsion does not correspond to Minkowski space–time; namely we do not have an acceptable generalization of energy from SR to flat theories with torsion, neither a criterion to determine whether the Equivalence Principle implies the full annihilation of the connection or, for example, only its symmetric Christoffel term. Torsion is important when dealing with more degrees of freedom than in GR, like Einstein–Cartan–Sciama–Kibble (ECSK) theory,[1] and when supersymmetries are required like in Supergravity and Superstring theories ($N = 1$ Supergravity is the low energy limit of Superstrings[8]). Supersymmetry also provides a reasonable explanation of some relevant cosmological features.[9]

Besides all the above considerations, an additional fact must be taken into account: even in Classical Mechanics, energy and momentum depend not only on the geometrical structure of the theory but also on the reference system in which these quantities are measured. An illustrative example is the case of accelerated observers in Minkowski space–time. In the scheme of Field Theory, we have shown that these kinds of observers will measure particle creation, depending on the parameters that characterize their acceleration[10] and also on the topological structure.[11]

With respect to the Equivalence Principle, we must note that, due to the annihilation of the gravitational "force" in free falling systems, the corresponding local gravitational energy and momentum densities must be also null. We should point out that when dealing with quantum mechanical states, not all freely falling frames are physically equivalent; this fact violates Local Position Invariance.[12] For example, in the case of a linear superposition of eigenstates with different masses (weak-flavor eigenstates of neutrinos), observable gravitational relative phases are induced as the redshift of a flavor oscillation clock; i.e. relative phases do not vanish along equipotential lines, even if the gravitational force does.[13] This effect, which is due to gravitation, does not vanish in free falling frames. Anyway, we will deal only with classical phenomena and work with classical matter in free falling systems satisfying certain local holomicity properties; later on, we discuss the corresponding trajectories, a classical concept (nonquantum) itself.

On the other hand, as it was shown in Ref. 7, the definition of total gravitational energy depends on asymptotic boundary conditions through surface terms. So we must use an approach that guarantees the local annihilation of the gravitational energy–momentum for any selection of the surface terms to be imposed.

Taking into account that the Lie derivative infinitesimal generators have the same geometrical structure of the Hamiltonian and the momentum and that those generators reduce to them in the special case of Minkowskian space–time, we will define energy–momentum via the introduction of a pseudovector related to the generators of the Lie derivatives. All this suggests that the concept of diffeomorphism is a very important tool in order to characterize physically conserved magnitudes in curved space–time. In fact, the Noether charge, defined for any diffeomorphic invariant Lagrangian, is used not only to define energy and momentum of physical systems, but to compute different kinds of physical magnitudes like the entropy of a black hole.[14] All this enables us to consider the Lie generators in temporal or spatial directions as the Hamiltonian and the momentum, respectively, not only for flat manifolds but also for any general metric theory.

We compare our results with those arising from other definitions of the Hamiltonian and momentum like the one corresponding to the ADM formalism and the usual definition of semiclassical theories used to define vacuum states. As the generators that we obtain are independent of the affine structure, we find that our energy and momentum definitions are independent of the connection of the manifold. Then, our results are valid in space–time with any connection, either with torsion or without torsion. Following these results, we are able to define an Equivalence Principle for non-Riemannian geometries that avoids known ambiguities and determines the characteristics of the dynamics of free falling particles.

The paper is organized as follows. In Sec. 2 the generators of the Lie derivative of the physical fields are found and surface terms are studied. In Sec. 3 the Lie generators are used to define an energy–momentum pseudovector density. In Sec. 4 the energy–momentum density pseudovector and the energy–momentum tensor are linked. In Sec. 5 we discuss the relation between reference systems and conserved quantities. Finally, in Sec. 6 we make a generalization of the Equivalence Principle from General Relativity (GR) to non-Riemannian manifolds.

2. Infinitesimal Generators of Diffeomorphism in General Manifolds

We suppose that the total action of the physical system is defined over a generic manifold assuming that the Lagrangian density contains only the first derivatives of the physical fields and eventually a position dependence. Superior order derivatives only may appear through a total divergence of a vectorial function F^μ. Then, in general, we have

$$W[\phi_a{}^\alpha, \partial_\mu\phi_a{}^\alpha, \partial_\mu\partial_\nu\phi_a{}^\alpha] = \int_{Z^4} \mathscr{L}_T(\phi_a{}^\alpha, \partial_\mu\phi_a{}^\alpha, \partial_\mu\partial_\nu\phi_a{}^\alpha)d^4x$$
$$= \int_{Z^4} \mathscr{L}(\phi_a{}^\alpha, \partial_\mu\phi_a{}^\alpha)d^4x$$
$$+ \int_{Z^4} \partial_\mu F^\mu(\phi_a{}^\alpha, \partial_\mu\phi_a{}^\alpha)d^4x\,, \tag{2.1}$$

where $Z^4 \subset U^4$, with U^4 the total manifold, d^4x the elementary four-volume in U^4 and $\phi_a{}^\alpha$ represents fields α with spin "a".

Now we take a family of one-parameter λ curves $\mathscr{C}(\lambda)$ with components $X^\mu(\lambda)$, $\lambda \in \mathbb{R}$, and the corresponding tangent vector field $u^\mu(x)$

$$u^\mu(x) = \frac{\partial X^\mu}{\partial\lambda}(\lambda)\,. \tag{2.2}$$

We define a generic diffeomorphism for the points of the manifold

$$x^\mu \to x^\mu - u^\mu(x)\Delta\lambda = x'^\mu \tag{2.3}$$

and the respective induced transformation over the fields:

$$\phi_a{}^\alpha(x) \to \phi'_a{}^\alpha(x')\,, \tag{2.4}$$

where $\phi'_a{}^\alpha(x')$ is the field $\phi_a{}^\alpha(x)$ transformed by diffeomorphism from the point x to the point x'.

These transformations change the action W into $W' = W + \Delta W$ and allows us to define the operation

$$D_u W \equiv \lim_{\Delta\lambda\to 0} \frac{\Delta W}{\Delta\lambda} = \int_{Z^4} \frac{\delta W}{\delta\phi_a{}^\alpha} \mathcal{L}_u\phi_a{}^\alpha d^4x + \mathrm{P}[u, \mathscr{L}, Z_+{}^3]$$
$$+ \int_{Z^4} \partial_\mu\left(\frac{\partial\partial_\tau F^\tau}{\partial\partial_\mu\partial_\sigma\phi_a{}^\alpha}\partial_\sigma\mathcal{L}_u\phi_a{}^\alpha - \partial_\sigma\frac{\partial\partial_\tau F^\tau}{\partial\partial_\mu\partial_\sigma\phi_a{}^\alpha}\mathcal{L}_u\phi_a{}^\alpha\right)d^4x\,, \tag{2.5}$$

where $\Delta W = W'[\phi'_a{}^\alpha] - W[\phi_a{}^\alpha]$, the Lie derivative $\mathcal{L}_u$ is

$$\mathcal{L}_u\phi_a{}^\alpha(x) \equiv \lim_{\Delta\lambda\to 0} \frac{\phi'_a{}^\alpha(x) - \phi_a{}^\alpha(x)}{\Delta\lambda}\,, \tag{2.6a}$$

where $\phi'_\alpha{}^a(x)$ is the result of the diffeomorphism over $\phi_\alpha{}^a(x')$ from x' to x, and

$$\mathbb{P}[u, \mathcal{L}, Z_+{}^3] = \int_{Z^3} \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_\alpha{}^a} \mathcal{L}_u \phi_\alpha{}^a - \mathcal{L} u^\mu \right) dS_\mu \,, \tag{2.6b}$$

where the vectorial density dS_μ is the elemental surface of ∂Z^4, and $\partial Z^4 = Z^3$. All the above computations are on shell: $\frac{\delta W}{\delta \phi_\alpha{}^a} = 0$.

Now we consider the submanifold $M^3 \subset U^4$ and define an atlas (X^i, X^0) by means of which M^3 is characterized by the condition $X^0 = \text{constant}$. As the metric tensor has not been yet defined, X^0 does not mean a time coordinate.

It is possible to extend the notion of Poisson bracket from flat space to U^4 as

$$[A(\phi_\alpha{}^a; \Pi_\alpha{}^a), B(\phi_\alpha{}^a; \Pi_\alpha{}^a)]_{\text{PB}} \equiv \int_{Z^3} \left(\frac{\delta A}{\delta \phi_\alpha{}^a} \frac{\delta B}{\delta \Pi_\alpha{}^a} - \frac{\delta A}{\delta \Pi_\alpha{}^a} \frac{\delta B}{\delta \phi_\alpha{}^a} \right) d^3X \,, \tag{2.7}$$

where $d^3X = dX^1\, dX^2\, dX^3$ is a scalar density of order -1; A and B functions of the fields $\phi_\alpha{}^a$ and $\Pi_\alpha{}^a$ with $\Pi_\alpha{}^a \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_\alpha{}^a)}$. With this definition, the Poisson brackets between quantity $\mathbb{P}$ (see (2.6b) and fields ϕ and Π, are

$$[\phi_\alpha{}^a; \mathbb{P}]_{\text{PB}} = \mathcal{L}_u \phi_\alpha{}^a \,, \tag{2.8a}$$

$$[\Pi_\alpha{}^a; \mathbb{P}]_{\text{PB}} = \mathcal{L}_u \Pi_\alpha{}^a \,. \tag{2.8b}$$

We conclude that the quantity, $\mathbb{P}$, is nothing but the infinitesimal generator of the Lie derivative. As no metric tensor was used up to now, it is not possible yet to define the Hamiltonian and the momentum generators.

3. Energy–Momentum Density Pseudovector

We want to interpret the infinitesimal generators as the Hamiltonian and the momentum of the system. We introduce the notion of space and time via the metric tensor $g_{\mu\nu}$ with signature $(-1, 1, 1, 1)$, and consider the curve $\mathscr{C}(\lambda)$, its tangent vector $\epsilon(x)$, with $x \in C(\lambda)$, and the orthogonal hypersurface M^3 to $\epsilon(x)$ at the point $\mathscr{C}(\lambda_0) = x_0$. If $\epsilon(x)$ is timelike, we can suppose that $\mathscr{C}(\lambda)$ is the trajectory of some observer. A generic "time" corresponding to this observer, can be defined as any real, continuous and increasing function of the proper time defined over $\mathscr{C}(\lambda)$. The local set of events that can be considered by this observer as simultaneous, are the points that belong to M^3 in the neighborhood of x_0.

Let us consider a set of curves $\mathscr{C}(\lambda)$. The corresponding tangent vectors are now a vectorial field $\epsilon(x)$. If it is possible to define the family of timelike hypersurfaces M^3 orthogonal to the field $\epsilon(x)$, with $x \in M^3$, then we are able to define a fluid of observers with velocity $\epsilon(x)$. Over these hypersurfaces we can define energy and momentum of this system of observers. So using $\epsilon \cdot \epsilon_T = 0$, with ϵ_T any vector that belongs to the tangent space to M^3, the generators $\mathbb{P}$:

$$\mathbb{P}[\epsilon, \mathcal{L}, M^3] = \mathcal{H} \,, \tag{3.1}$$

$$\mathbb{P}[\epsilon_T, \mathcal{L}, M^3] = P \,, \tag{3.2}$$

are the "Hamiltonian" and the "momentum" of the physical system described by the Lagrangian $\mathscr{L}$ measured in the reference system in which the observers have four-velocity $\epsilon(x)$. This is so because (3.1) and (3.2) are the infinitesimal generators of the time displacement $\epsilon(x)$ and the space displacement $\epsilon_T(x)$ respectively. In this sense, definition (3.1) and (3.2) are the natural extensions of the concepts of Hamiltonian and momentum from flat space–time to curved space–time. These definitions depend on the metric tensor via the timelike and spacelike vectors $\epsilon(x)$ and $\epsilon_T(x)$, but do not depend on the affine connection which is not even defined. If the Lagrangian does not have an explicit dependence on time, then, our Hamiltonian can be considered the energy of the system.

Now, from (2.6b), we define the pseudovector

$$T^\mu(u) = \frac{\partial \mathscr{L}}{\partial \partial_\mu {\phi_\alpha}^a} \mathcal{L}_u {\phi_\alpha}^a - \mathscr{L} u^\mu , \tag{3.3}$$

where u^μ is any vectorial field. No spin "information" will remain since diffeomorphisms only depend on spatial transformation from point x to point x'. If $u = \epsilon$, then $(g)^{-1/2}T^\mu(u)$ represents the energy density flux; if $u = \epsilon_T$, $(g)^{-1/2}T^\mu(u)$ represents the momentum density flux. In both cases these quantities are measured by observers (free falling or not) with four-velocity ϵ. We remark that T^μ represents energy or momentum density if vector u is timelike or spacelike respectively, although the coordinate index "μ" is a temporal or spatial index.

We note that if in the reference system $\{X^\mu\}$, T^μ satisfies

$$\partial_\nu T^\nu = 0 \tag{3.4}$$

at any point of Z^4. Then the generator $\mathbb{P}$:

$$\mathbb{P}[u, \mathscr{L}, Z^3] = \int_{Z^3} (g)^{-1/2} T^\mu \, dS_\mu$$

is a conserved quantity.

If (3.4) is a local property, valid only at a specific $x = x_0$, then quantity $\mathbb{P}$ is not conserved. Anyway, the total flux of T^μ over a closed elementary volume containing x_0 is zero.[15] For this reason this property can be called the "local energy–momentum conservation."

4. The Energy–Momentum Density Pseudovectors

If we choose the scalar curvature as the gravitational Lagrangian, we can separate the total action of the system W into W_M and W_G, where W_M includes the free action of the other fields ${\phi_\alpha}^a$ (matter fields, torsion) and the interaction between matter–matter and matter–geometry, while W_G corresponds to the kinetic term of the metric field $g_{\mu\nu}$. The total action becomes

$$W[g_{\mu\nu}, {\phi_\alpha}^a] = W_M[g_{\mu\nu}, {\phi_\alpha}^a] + W_G[g_{\mu\nu}] \tag{4.1}$$

with $W_M = \int_{Z^4} \mathscr{L}_M \, d^4x$ and $W_G = \int_{Z^4} \mathscr{L}_{TG} \, d^4x$.

The total gravitational Lagrangian is

$$\mathscr{L}_{\mathrm{TG}} = g^{1/2}R = \mathscr{L}_g + \partial_\mu F^\mu\,, \tag{4.2}$$

where

$$\mathscr{L}_G = g^{1/2}g^{\rho\sigma}\left(\left\{ {\mu \atop \rho\lambda} \right\}\left\{ {\lambda \atop \sigma\mu} \right\} - \left\{ {\mu \atop \rho\sigma} \right\}\left\{ {\lambda \atop \mu\lambda} \right\}\right)$$

and

$$F^\mu = g^{1/2}\left(g^{\rho\lambda}\left\{ {\mu \atop \rho\lambda} \right\} - g^{\mu\rho}\left\{ {\lambda \atop \rho\lambda} \right\}\right).$$

Some comments are in order. The Lagrangian $\mathscr{L}_G$ of (4.2) comes from splitting the curvature tensor $R(\Gamma)$ according to Ref. 16. We consider the gravitational Lagrangian as the scalar curvature built up with the Christoffel symbols depending only on the metric and its derivatives. This is a usual procedure when the action is a linear functional of the scalar curvature field. In this case the nonmetric terms (like torsion depending terms) of the connection, may be incorporated into the matter action [W_M in (4.1)]. The motivation to include the contributions of torsion terms (and other non-Riemannian parts of the connection) in the matter Lagrangian, is to represent gravitation only by the metric tensor field $g_{\mu\nu}$. This procedure is used, for example, in ECSK theory[1] where the relation between torsion and matter is algebraic via an equation which relates torsion with intrinsic angular momentum. In the case of Supergravity the procedure of splitting the curvature tensor is also used.[17] For example, in Ref. 9 we coupled the non-Riemannian connection as a part of cosmological matter. In the case of $N = 1$ Supergravity, there exist a supersymmetry that transforms metric into gravitino ψ, while torsion S is related with this last field via: $S_{\mu\nu}{}^\alpha = \psi_\mu\gamma^\alpha\psi_\nu$. Here the incorporation of torsion in the matter Lagrangian is, strictly speaking, merely a question of notation, because torsion and metric transform one into the other via supersymmetries.

On the other hand, the addition or the subtraction of a total time derivative in the Lagrangian or a total divergence in a Lagrangian density, leads to a new definition of the momentum and of the infinitesimal generators. As these transformations do not change the dynamical equations, the new generators and momentums are canonical transformations of the old ones. This is not the case of theories with constraints. In fact, in these theories, a divergence cannot necessarily be discarded since such a term may cease to be a divergence upon elimination of constraints. It is only possible to subtract total derivatives in the Lagrangian if the constraints are incorporated as suitable Lagrange multipliers. As was pointed in Ref. 18, this is the case of GR. In that article it was shown that some of the field equations can be considered as constraints caused by Lagrange multipliers. For this reason, in GR the subtraction of total derivatives in the Lagrangian can be considered as a canonical transformation of the action. For example, in Ref. 19 the total divergence subtraction leads, precisely, to the Lagrangian (4.2). Another example of subtraction of a divergence is shown in Ref. 20. This procedure can be extended

to other gravitational theories that include the scalar curvature in the definition of the gravitational action like ESCK (see Ref. 1, p. 400 and its appendix).

Now we return to the matter action. If W_M is invariant under diffeomorphisms, then it must satisfy

$$D_u W_M = 0\,, \tag{4.3}$$

where D_u is the operation defined in (2.5).

As a consequence of (4.3), we have that the tensor $T^{\mu\nu}$ defined as

$$T^{\mu\nu} \equiv \frac{1}{g^{1/2}}\left(\frac{\partial \mathscr{L}_M}{\partial \partial_\mu \phi_\alpha{}^a} \overset{\{\}}{\nabla}{}^\nu \phi_\alpha{}^a - \mathscr{L}_M g^{\mu\nu}\right) \tag{4.4}$$

satisfies

$$T^{\mu\nu} = -\frac{2}{g^{1/2}}\frac{\delta W_M}{\delta g_{\mu\nu}} \tag{4.5}$$

and

$$\overset{\{\}}{\nabla}_\mu T^{\mu\nu} = 0\,. \tag{4.6}$$

Definition (4.4) corresponds to the metric energy–momentum tensor, and $\overset{\{\}}{\nabla}_\mu$ is the "covariant derivative" corresponding to a tensor of any rank built up with the Christoffel symbols.

Using (3.3) and (4.4)–(4.6), and after a straightforward computation, we obtain that the four-momentum density (see (3.3)) corresponding to the action W_M is

$$T_M{}^\nu = g^{1/2}\epsilon_\sigma T^{\nu\sigma}\,, \tag{4.7}$$

where $T_M{}^\nu$ is given by (3.3) if $\mathscr{L}$ is the matter Lagrangian $\mathscr{L}_M$ and u is ϵ. As relation (4.7) was obtained without any reference to the affine structure, the result is valid in any metric manifold. On the other hand, we note that if the manifold is endowed with an arbitrary definition of parallel transport (metric or not, symmetric or not), the results remain valid and the metric energy–momentum tensor built up with a Christoffel connection, in general, is not the covariant version of the SR $T^{\mu\nu}$ (i.e. ordinary derivatives replaced by covariant derivatives in Eq. (4.4).

At this point, it is interesting to compare our results with classical definitions of the Hamiltonian and the momentum in curved space–time. In the ADM formalism, the Hamiltonian is obtained in metric differential manifolds where it is possible to define spatial surfaces Z^3 with coordinates x^i $(i = 1, 2, 3)$ labeled by a "temporal" parameter t $(3+1$ formalism). The gravitational action is the Hilbert–Einstein one. Under these conditions, using the definition of the surface term given in (4.2), and taking into account the scalar structure of the Hilbert–Einstein action, in (2.5), quantity

$$\mathbb{P}[u, \mathscr{L}, Z_+{}^3] + \int_{Z^3}\left(\frac{\partial\partial_\tau F^\tau}{\partial\partial_\mu\partial_\sigma\phi_\alpha{}^a}\,\partial_\sigma \mathcal{L}_u\phi_\alpha{}^a - \partial_\sigma\frac{\partial\partial_\tau F^\tau}{\partial\partial_\mu\partial_\sigma\phi_\alpha{}^a}\,\mathcal{L}_u\phi_\alpha{}^a\right)dS_\mu \tag{4.8}$$

is conserved and may be interpreted as the gravitational energy. Writing (4.8) in terms of the metric tensor, we have

$$E_G = \int g^{ij}(g^{kp}g_{kj,p} - g^{kp}g_{kp,j})dS_i\,, \tag{4.9}$$

where dS_i is the two-dimensional surface element at spatial infinity. In a linearized theory with a flat background, last equation becomes:

$$E_G = \int (h_{ki,k} - h_{qq,i})dS_i\,, \tag{4.10}$$

where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with η the Minkowski metric; and we do not distinguish between upper and lower indices on S_i. Equation (4.10) coincides with the standard ADM energy definition obtained for example in Ref. 21.

On the other hand, with the same procedure, we are able to obtain the ADM Hamiltonian of the matter fields:

$$\mathbb{P}[\epsilon, \mathscr{L}_M, Z^3] = \int_{Z^3} T^{0\nu}\epsilon_\nu (g)^{1/2}\, d^3x\,. \tag{4.11}$$

We remark that our procedure allows us to extend the validity of (4.9) and (4.10) to generic non-Riemannian space–times.

We note that (4.7) not only establishes the relation between the four-momentum pseudotensor density flux and the metric energy–momentum tensor, in fact, it is a justification of the definition introduced in Ref. 22:

$$\mathcal{H} \equiv \int_{Z^3} T^{\mu\nu}\epsilon_\nu\, dS_\mu \tag{4.12}$$

which is employed in semiclassical theories in order to define the vacuum state.

Finally, we would like to discuss how to compare our "Christoffel" definition of energy with a "complete connection" definition of energy. From (4.12), we see that in the reference system where $\epsilon = (1,0,0,0)$, $\mathcal{H}$ is the energy E:

$$E = \int_\Sigma T^\nu\, d\Sigma_\nu = \int_\Sigma T^{\nu 0}\, d\Sigma_\nu\,, \tag{4.13}$$

where Σ is a spatial hypersurface, T^ν is defined in (3.3) and $T^{\nu 0}$ is the component $\nu 0$ of the energy–momentum tensor defined with respect to the Christoffel symbols (4.4). An alternative approach could be

$$E' = \int_\Sigma \overset{\Gamma}{T}{}^{\nu 0}\, d\Sigma_\nu \tag{4.14}$$

with $\overset{\Gamma}{T}{}^{\nu 0}$ the component $\nu 0$ of the energy–momentum tensor defined with respect to the complete connection:

$$\overset{\Gamma}{T}{}^{\mu\nu} \equiv \frac{1}{g^{1/2}}\left(\frac{\partial \mathscr{L}_M}{\partial \partial_\mu \phi_\alpha{}^a}\, \overset{\Gamma}{\nabla}{}^\nu \phi_\alpha{}^a - \mathscr{L}_M g^{\mu\nu}\right). \tag{4.15}$$

The difference between these two definitions of energy depends on $\Delta T^{\nu 0} = \overset{\Gamma}{T}{}^{\nu 0} - T^{\nu 0}$. For the case of ESCK theory in U_4, $\Delta T^{\nu 0}$ could be obtained from Eqs. (3.8) and (3.10) of Ref. 1 (p. 399).

Our reasons for choosing (4.13) instead of (4.14), will be explained in Sec. 6.

5. Global Conservations

Taking the covariant divergence defined through the Christoffel connection of expression (4.7), we see that

$$\overset{\{\}}{\nabla}_\mu (g^{1/2} T_M{}^\mu) = T^{\mu\sigma} \overset{\{\}}{\nabla}_\mu \epsilon_\sigma = T^{\mu\sigma} \mathcal{L}_\epsilon g_{\mu\sigma} \tag{5.1}$$

which in general is different from zero. The fundamental role of the observers in the definition of energy–momentum can be seen in one example in Minkowski space–time. In this space–time, if ϵ is the tangent vector to the temporal global Lorentzian coordinate, then the corresponding observers are inertial (nonaccelerated) observers. As in this case $\mathcal{L}_\epsilon g_{\mu\sigma} = 0$, both energy and momentum are globally conserved. But if we consider general accelerated observers, the Lie derivative of the metric tensor is, in general, different from zero, which implies that energy:

$$\mathcal{H} = \int_{Z^3} T^{\mu\nu} \epsilon_\nu \, dS_\mu \tag{5.2}$$

and momentum:

$$P = \int_{Z^3} T^{\mu\nu} \epsilon_{T\nu} \, dS_\mu \tag{5.3}$$

will not be conserved. But in flat space–time there is an interesting example for which accelerated observers may define a conserved Hamiltonian. If we consider Rindler coordinates:

$$ds^2 = e^{2\xi}(d\xi^2 - d\eta^2) , \tag{5.4}$$

where ξ and η are the spatial and temporal coordinates respectively, then Rindler observers are those whose trajectories follow curves η. We see that $\mathcal{L}_\eta g_{\mu\nu} = 0$; then for these observers, energy (5.2) defined on the hypersurface $\eta = cte$ is conserved. Nevertheless, as $\mathcal{L}_\xi g_{\mu\nu}$ is not zero, momentum is not conserved. So we note that even in flat space–time, conservation of energy and momentum depend on the acceleration of the observers. At this point, we should note that in those curved space–times, such as Robertson–Walker, where it would be possible to define Killing temporal observers, we have the analogous situation ($\mathcal{L}_k g_{\mu\nu} = 0$ with k^μ the four-velocity of the observers), and the Hamiltonian can be well defined. This is the case for de Sitter and anti-de Sitter space treated for example in Ref. 4. In that article the authors have defined the quantity $T^{\mu\nu} k_\nu$ as the Killing energy–density — k_ν being a Killing vector — and found that $\int T^{0\nu} k_\nu \, d^3x$ is a conserved energy. We must note that in our approach, both the interpretation of quantity $T^{\mu\nu} k_\nu$ as the energy–density and its conservation, are natural consequences of our formalism.

6. Equivalence Principle and Local Conservation

In this section, we deal with more general space–times where global conditions are not necessarily fulfilled and focus our attention on local properties in order to introduce magnitudes that can be locally conserved.

In any nonsingular point x_0 of a Riemannian geometry we can define a local Lorentzian free falling reference system by means of the tetrad $\hat{e}_A(x_0)$ ($A, B, \ldots = 0, 1, 2, 3$; $\hat{e}_A \cdot \hat{e}_B = \eta_{AB}$ the Minkowski metric). The local annihilation of the connection corresponding to this anholomic tetrad, can be called the *local principle of holonomy*:

$$\mathcal{L}_{\hat{e}_A}\hat{e}_B(x_0) = 0\,, \quad \text{or equivalently} \quad \mathcal{L}_{\hat{e}_A} g_{\mu\nu}(x_0) = 0\,. \tag{6.1}$$

Now we consider the total energy–momentum pseudovector defined in (3.3). For the gravitational field, due to the second order derivatives in the Lagrangian, we must add to the energy–momentum pseudovector, the surface terms contributions. Then the corresponding pseudovector is

$$\begin{aligned} T_{\mathrm{TG}}{}^{\mu}(u = \hat{e}_A) = {} & \frac{\partial \mathscr{L}_g}{\partial \partial_\mu g_{\nu\sigma}} \mathcal{L}_{\hat{e}_A} g_{\nu\sigma} - \mathscr{L}_g e_A{}^{\mu} + \frac{\partial \partial_\tau F^\tau}{\partial \partial_\mu g_{\nu\delta}} \mathcal{L}_u g_{\nu\delta} - \partial_\tau F^\tau e_A{}^{\mu} \\ & + \frac{\partial \partial_\tau F^\tau}{\partial \partial_\mu \partial_\sigma g_{\nu\delta}} \partial_\sigma \mathcal{L}_u g_{\nu\delta} - \partial_\sigma \frac{\partial \partial_\tau F^\tau}{\partial \partial_\mu \partial_\sigma g_{\nu\delta}} \mathcal{L}_u g_{\nu\delta} \end{aligned} \tag{6.2}$$

(with $\mathscr{L}_g$ given by (4.1)); while the "matter" energy–momentum pseudovector reads:

$$T_M{}^{\mu}(u = \hat{e}_A) = \frac{\partial \mathscr{L}_M}{\partial \partial_\mu \phi_\alpha{}^a} \mathcal{L}_{\hat{e}_A} \phi_\alpha{}^a - \mathscr{L}_M e_A{}^{\mu}\,. \tag{6.3}$$

In the free-falling systems satisfying (6.1) the following quantities: $\mathscr{L}_g$, $\mathcal{L}_{\hat{e}_A} g_{\mu\nu}$, $\partial_\sigma \mathcal{L}_{\hat{e}_A} g_{\mu\nu}$, F^τ and $\partial_\tau F^\tau$ are identically zero. Then in this reference system we have that the energy–momentum densities satisfy the following relations:

$$T_g{}^{\mu}(\hat{e}_A)(x_0) = 0\,, \tag{6.4a}$$

$$\overset{\{\}}{\nabla}_\mu (g^{-1/2} T_g{}^{\mu}(\hat{e}_A))(x_0) = 0\,, \tag{6.4b}$$

$$\overset{\{\}}{\nabla}_\mu (g^{-1/2} T_M{}^{\mu}(\hat{e}_A))(x_0) = 0\,. \tag{6.4c}$$

So we conclude that, in a Riemannian geometry, the energy–momentum flux of the gravitational and matter fields, and the gravitational energy–density, are locally zero when they are measured in the free falling reference system.

But, if in analogy with the Riemannian case, in a non-Riemannian geometry we demand the vanishing of local free falling tetrad connection ($\Gamma^C_{AB} = 0$) like in Ref. 23, we obtain, in general, that

$$\overset{\{\}}{\nabla}_\mu (g^{-1/2} T_M{}^{\mu})(x_0) \neq 0 \tag{6.5}$$

and therefore conservation is lost. Of course this is not a desirable property of a free falling system. In a Riemannian case, not only the affine connection can be taken to be null but also local holonomicity is satisfied and this fact guarantees the local energy–momentum conservation (6.4b) and (6.4c). But, in general, the vanishing of the tetrad connection does not imply the vanishing of the Lie derivative of the tetrad. In this case

$$\Gamma^{C}_{AB} = 0 \to \mathcal{L}_{\hat{e}_B}\hat{e}_A = S(\hat{e}_B, \hat{e}_A)\,, \tag{6.6}$$

where S is the torsion.

Taking into account the above discussion, we propose another extension of the Equivalence Principle to the non-Riemannian case: the local holonomicity property (6.1). This property does not imply a locally null connection, but guarantees local energy–momentum conservation. Then our Equivalence Principle will be related to reference systems where

$$\mathcal{L}_{\hat{e}_B}\hat{e}_A = 0 \tag{6.7}$$

which now implies that the physical properties (6.4) are fulfilled. The properties represent classical behaviors of matter; our Equivalence Principle, related to condition (6.7), has nothing to do with quantum effects as discussed in the introduction. On the other hand, if we start from our Equivalence Principle version, we obtain, after a straightforward computation, that trajectories of free falling particles are geodesics:

$$u^{\mu}\overset{\{\}}{\nabla}_{\mu}u^{\nu} = 0\,, \tag{6.8}$$

where u^{μ} is the tangent vector to the free falling trajectory. Conversely if trajectories are geodesic, the local energy–momentum flux is null. For example in a perfect fluid without pressure with energy-density ρ and four-velocity u^{μ}, the energy–momentum reads

$$T^{\mu\nu} = \rho u^{\mu}u^{\nu} \tag{6.9}$$

making zero the local energy–momentum flux:

$$\overset{\{\}}{\nabla}_{\mu}(g^{1/2}T_{M}{}^{\mu}) = T^{\mu\nu}\overset{\{\}}{\nabla}_{\mu}u_{\nu} = \rho u^{\mu}u^{\nu}\overset{\{\}}{\nabla}_{\mu}u_{\nu} = 0 \tag{6.10}$$

due to (6.8).

Instead, if free falling particles were related to the Equivalence Principle based on a local vanishing of the connection Γ^{C}_{AB}, they would follow autoparallel trajectories:

$$u^{\mu}\overset{\Gamma}{\nabla}_{\mu}u^{\nu} = 0\,. \tag{6.11}$$

So, in our version, particles with conserved energy and momentum are those which follow geodesics.

7. Conclusions

Integrating (6.2), we are able to introduce a gravitational Hamiltonian that conduces, with adequate asymptotic conditions, to the standard definition of gravitational energy. With our formalism we also justify the usual definition of matter Hamiltonian (5.2) and momentum (5.3) in curved space–time. We extend these definitions to manifolds with torsion which admit global foliation. In all these definitions, the role of observers appears explicitly via the vectorial parameter ϵ.

The possibility of defining conserved Hamiltonians for accelerated observers in flat space–times, can also be extended to semiclassical theories justifying the existence of Fock spaces with well-defined vacuum in both flat (Rindler) and curved space–times with temporal Killing vectors. Then, particle creation can be linked to the energy of the moving observers.[10]

Introduction of the energy–momentum pseudovector solves some ambiguities in the definition of local energy–momentum. We use precisely this pseudovector in order to introduce our extension of the Equivalence Principle to non-Riemannian geometries, discussing the structure of the surface terms. We showed that this principle predicts, for any case, geodesic motion for free falling particles. Then, any deviation from geodesic trajectory can be attributed to nonconservative behavior. This property can be applied to study physical phenomena like orbiting systems and link accelerated observers with observable parameters such as orbital period and energy decay. In a forthcoming paper, we will use our definition of the Equivalence Principle in order to relate a nonconservative behavior of astrophysical systems to non-Riemannian structures of space–time. This would be the case of strong gravitational fields generated in a neighborhood of neutron or black holes, whose entropy is the Noether charge corresponding to diffeomorphic transformation.[24]

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